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Reversible transformations of shapes and sizes of uncharged ergospheres

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Abstract. The shape and size of a Kerr ergosphere and their reversible transformations are studied systematically. It is proved that due to the ergosphere's total angular momentum, a bulge always forms on its outer boundary either on the axis of rotational symmetry or off it. During the reversible increase of the angular momentum, because of the injection of particles into the ergosphere, the bulge can either approach the symmetry plane or recede from it, while its angular separation from the positive rotation axis increases continuously, approaching the value of 30° for the extreme Kerr ergosphere. The reversible changes of the ergosphere's linear dimensions parallel to the rotation axis and perpendicular to it are studied and the corresponding ranges of their permitted values are specified.

1. Introduction

The stationary-limit surface and the exterior event horizon of a Kerr black hole of total mass energy m and total angular momentum S (in geometrised units) forming the outer and the inner boundary, respectively, of the hole's ergosphere, in Boyer–Lindquist (1967) coordinates t, r, ϑ, φ ($-\infty < t < \infty$, $r_+ < r < \infty$, $0 \leqslant \vartheta \leqslant \pi$, $0 \leqslant \varphi \leqslant 2\pi$) are located at distances R_+ and r_+ defined by

$$r = R_+ \equiv m\left[1 + \left(1 - \frac{S^2}{m^4}\cos^2\vartheta\right)^{1/2}\right] \tag{1.1}$$

$$r = r_+ \equiv m\left[1 + \left(1 - \frac{S^2}{m^4}\right)^{1/2}\right]. \tag{1.2}$$

In terms of r, ϑ the Kerr coordinates x, y, z are expressed through the relations, suitable for our purposes,

$$z = r\cos\vartheta, \qquad \rho \equiv (x^2 + y^2)^{1/2} = (r^2 + S^2/m^2)^{1/2}\sin\vartheta \tag{1.3}$$

and

$$\frac{\rho^2}{r^2 + S^2/m^2} + \frac{z^2}{r^2} = 1. \tag{1.4}$$

If x, y, z are interpreted as orthogonal Cartesian coordinates, r, ϑ, φ will be interpreted as quasi-spheroidal coordinates, and in the plane (ρ, z) the surfaces of constant r are confocal ellipsoids of revolution around the z axis with semiaxes $(r^2 + S^2/m^2)^{1/2}$ and r. In the case of the ergosphere's inner boundary r is independent of ϑ. This is not the case for the ergosphere's outer boundary on which r depends on ϑ. Moreover, since z has the same sign as $\cos\vartheta$, the surface of constant ϑ is a half-hyperboloid, confocal to the ellipsoid, truncated at its waist, lying in the half-space $z \gtrless 0$ according to $\vartheta \lesseqgtr \frac{1}{2}\pi$, and can be considered as measuring angular separations from the z axis.

In § 2 of this article we study the shape of a Kerr ergosphere in terms of its parameters m and S and we prove that due to the angular momentum of the ergosphere a bulge always forms on its outer boundary. The behaviour of this bulge under the reversible transformations of m and S is studied in § 3. This permits us to study the behaviour of the ergosphere's linear dimensions in § 4. Finally, in § 5 the results of the previous sections are discussed and compared with analogous results of earlier authors.

2. The shape of the stationary-limit-surface

It is straightforward to verify that, if the cosmic censorship hypothesis is valid ($m^2 > S$), the function $z(\vartheta)$ defined by equation (1.3), and evaluated on the stationary-limit surface, presents extrema ($z' = 0$; a prime denotes total derivative with respect to ϑ, and a dot will denote total derivative with respect to S), when

$$\sin\vartheta = 0 \tag{2.1}$$

and/or

$$1 + \left(1 - \frac{S^2}{m^4}\cos^2\vartheta\right)^{1/2} - \frac{S^2}{m^4}\cos^2\vartheta\left(1 - \frac{S^2}{m^4}\cos^2\vartheta\right)^{-1/2} = 0. \tag{2.2}$$

These two conditions are independent of each other, but they can hold simultaneously, if

$$S/m^2 = \sqrt{3}/2. \tag{2.3}$$

Direct consequences of equation (2.3) and the standard formula due to Christodoulou (1970)

$$m^2 = m_{\text{ir}}^2 + S^2/4m_{\text{ir}}^2 \tag{2.4}$$

where m_{ir} is the ergosphere's irreducible mass, are the relations

$$m/m_{\text{ir}} = 2/\sqrt{3} \qquad \text{and} \qquad S/m = m_{\text{ir}}. \tag{2.5}$$

Now a simple calculation shows that, if equation (2.1) is true,

$$z'' \gtrless 0 \quad \text{if} \begin{cases} S/m \gtrless m_{\text{ir}} & \text{for } \vartheta = 0 \qquad (2.6a) \\ S/m \lesseqgtr m_{\text{ir}} & \text{for } \vartheta = \pi. \qquad (2.6b) \end{cases}$$

According to equations (2.6a) the extreme value of z on the axis $z > 0$ will be maximum (minimum), if the angular momentum per unit mass is smaller (larger) than the irreducible mass. The interpretation of equations (2.6b), where $z < 0$, mirrors that of equations (2.6a). In all these cases the extremum, Z, of z is

$$Z = \begin{cases} +r_+ & \text{for } \vartheta = 0 \\ -r_+ & \text{for } \vartheta = \pi. \end{cases} \tag{2.7}$$

Next, if equation (2.2) is valid, then

$$\cos\vartheta = \pm\frac{\sqrt{3}}{2}\frac{m^2}{S} = \pm\frac{\sqrt{3}}{2}\left(\frac{m_{\text{ir}}^2}{S} + \frac{S}{4m_{\text{ir}}^2}\right) \qquad \text{for } \vartheta \lesseqgtr \tfrac{1}{2}\pi \tag{2.8}$$

and

$$z'' \lesseqgtr 0 \qquad \text{if } \vartheta \lesseqgtr \tfrac{1}{2}\pi. \tag{2.9}$$

Therefore the solution (2.8) determines the position of a maximum (for $\vartheta < \frac{1}{2}\pi$) or minimum (for $\vartheta > \frac{1}{2}\pi$) of z at a point off the rotation axis. Notice that this solution is consistent with equations (2.6) valid for points on the rotation axis, and moreover that ϑ is determined solely in terms of the hole's parameters. The extremum in z is

$$Z = \pm \tfrac{3}{4}\sqrt{3}\, m^3/S \qquad \text{for } \vartheta \lesseqgtr \tfrac{1}{2}\pi \tag{2.10}$$

and occurs at a distance

$$r = \tfrac{3}{2}m > |Z|. \tag{2.11}$$

Similarly we verify that ρ always attains its maximal value

$$R = (m^2 + S^2/m^2)^{1/2} \tag{2.12}$$

on the symmetry plane.

The existence of a maximum in the value of z (>0) means that at the corresponding position a bulge forms on the ergosphere's outer boundary. If $S/m < m_{\mathrm{ir}}$ the bulge forms on the rotation axis at the distance of the event horizon, while if $S/m > m_{\mathrm{ir}}$ the bulge's position is off the axis and is determined solely in terms of m and S.

3. Reversible transformations of the ergosphere's shape

In this section we study the changes of the shape of the ergosphere's boundaries, due to reversible increments, $\mathrm{d}m$ and $\mathrm{d}S$, of its parameters, caused by the injection of particles into the ergosphere. These changes have to be related (Christodoulou 1970) by

$$\mathrm{d}m_{\mathrm{ir}} \geqslant 0 \qquad \mathrm{d}m \geqslant S\,\mathrm{d}S/m4m_{\mathrm{ir}}^2 \tag{3.1}$$

with the equality (inequality) sign valid for reversible (irreversible) transformations. The attainable range of the ergosphere's reversible transformations is described by

$$\mathrm{d}m_{\mathrm{ir}} = 0, \qquad 0 \leqslant S \leqslant m^2, \qquad 1 \leqslant m/m_{\mathrm{ir}} \leqslant \sqrt{2} \tag{3.2}$$

where the quantities m, S and m_{ir} describe the system of the ergosphere and the particles.

3.1. The behaviour of ϑ

With the aid of equations (2.8), (3.1) and (3.2) we find

$$\dot{\vartheta} = \pm \frac{\sqrt{3}}{2\sin\vartheta}\left(\frac{m_{\mathrm{ir}}^2}{S^2} - \frac{1}{4m_{\mathrm{ir}}^2}\right) \qquad \text{for } \vartheta \lesseqgtr \tfrac{1}{2}\pi \tag{3.3}$$

and

$$\ddot{\vartheta} = \mp\left[\frac{\sqrt{3}}{\sin\vartheta}\frac{m_{\mathrm{ir}}^2}{S^3} + \frac{3\sqrt{3}}{8\sin^3\vartheta}\left(\frac{m_{\mathrm{ir}}^2}{S} + \frac{S}{4m_{\mathrm{ir}}^2}\right)\left(\frac{m_{\mathrm{ir}}^2}{S^2} - \frac{1}{4m_{\mathrm{ir}}^2}\right)\right] \qquad \text{for } \vartheta \lesseqgtr \tfrac{1}{2}\pi \tag{3.4}$$

where a dot denotes total derivative with respect to S, keeping m_{ir} constant. Thus for all the possible reversible transformations, on the one hand

$$\dot{\vartheta} \gtreqless 0 \qquad \text{for } \vartheta \lesseqgtr \tfrac{1}{2}\pi \tag{3.5}$$

with the equality sign valid for the extreme Kerr ergosphere

$$S = m^2 = 2m_{\mathrm{ir}}^2 \tag{3.6}$$

and on the other hand

$$\ddot{\vartheta} \gtrless 0 \qquad \text{for } \vartheta \gtrless \tfrac{1}{2}\pi. \tag{3.7}$$

Therefore for the entire attainable range of reversible transformations ϑ, if smaller than $\frac{1}{2}\pi$, increases with increasing S (and decreases with decreasing S), while, if larger than $\frac{1}{2}\pi$, it decreases with increasing S (and increases with decreasing S). This means that, as S increases from mm_{ir}, the angular separation of the bulge from the rotation axis increases continuously. The extreme value, Θ, of ϑ occurs for the extreme Kerr ergosphere, and, as deduced from equation (2.8),

$$\Theta = 30°, 150° \qquad \text{for } \vartheta \lessgtr \tfrac{1}{2}\pi. \tag{3.8}$$

The corresponding values of $|z|$ and r are $\frac{3}{4}\sqrt{6}m_{\mathrm{ir}}$ and $\frac{3}{2}\sqrt{2}m_{\mathrm{ir}}$, respectively. Notice that the above conclusions are independent of the specific way with which S increases (reversibly).

3.2. *The behaviour of Z*

When the bulge forms on the rotation axis ($0 < S \leqslant mm_{\mathrm{ir}}$),

$$Z = \pm r_+ \tag{3.9}$$

and hence the behaviour of the bulge's distance parallel to the rotation axis from the symmetry plane is the same as that of the event horizon distance. It is straightforward to prove that, if equations (3.2) are valid,

$$\pm\dot{Z} < 0 \qquad \text{for } \vartheta \lessgtr \tfrac{1}{2}\pi. \tag{3.10}$$

Considering for simplicity the case $\vartheta < \frac{1}{2}\pi$ we observe that Z decreases continuously, as S increases. For S almost equal to zero the bulge is at a maximal distance (for $z > 0$)

$$Z_{\max} = 2m = 2m_{\mathrm{ir}} \tag{3.11}$$

from the symmetry plane, and with increasing S it decreases, attaining for $S = mm_{\mathrm{ir}}$ a limiting value

$$Z_{\mathrm{lim}} = \tfrac{3}{2}m = \sqrt{3}m_{\mathrm{ir}}. \tag{3.12}$$

It is remarkable that at this limiting position

$$\dot{Z} = -3/8m_{\mathrm{ir}} < 0. \tag{3.13}$$

In view of the symmetry with respect to the equatorial plane the case $\vartheta > \frac{1}{2}\pi$ ($Z < 0$) eactly mirrors the one just discussed.

After that limiting position, and as S increases ($S > mm_{\text{ir}}$), the bulge moves off the rotation axis, at a distance (see equation (2.10))

$$Z = \pm \frac{3\sqrt{3}}{4} \frac{m^3}{S} (\vartheta \lessgtr \tfrac{1}{2}\pi); \qquad r = \tfrac{3}{2} m \tag{3.14}$$

where equations (1.3) and (2.8) have been used. From equations (3.14) we verify that

$$\pm \dot{Z} < 0 \text{ (for } \vartheta \lessgtr \tfrac{1}{2}\pi) \qquad \text{if } mm_{\text{ir}} < S < \sqrt{2} m_{\text{ir}}^2 \tag{3.15a}$$

$$\dot{Z} = 0 \qquad \text{if } S = \sqrt{2} m_{\text{ir}}^2 \tag{3.15b}$$

$$\pm \dot{Z} > 0 \text{ (for } \vartheta \lessgtr \tfrac{1}{2}\pi) \qquad \text{if } \sqrt{2} m_{\text{ir}}^2 < S < 2 m_{\text{ir}}^2 \tag{3.15c}$$

and so, as S increases reversibly, the bulge off this axis can either approach the symmetry plane or recede from it in contrast to the uniform behaviour of the angular separation ϑ and of the extremum Z on the rotation axis. Thus in the case $\vartheta < \frac{1}{2}\pi$ for S slightly larger than mm_{ir}, ϑ increases from zero. Since, however, at this region $\dot{Z} < 0$ (see equation (3.13)), Z initially decreases with increasing S, until, for $S = \sqrt{2} m_{\text{ir}}^2$ (whence $\dot{Z} = 0$, $\ddot{Z} > 0$), it is at its closest distance to the symmetry plane, off the rotation axis,

$$Z_{\text{c}} = \frac{3^{3/2}}{2^{5/2}} \frac{m^3}{m_{\text{ir}}^2} = \frac{27}{16} m_{\text{ir}} \tag{3.16a}$$

at

$$r_{\text{c}} = (\tfrac{3}{2})^{3/2} m_{\text{ir}} = \frac{2^{5/2}}{3^{3/2}} Z_{\text{c}} \tag{3.16b}$$

and an angular separation ϑ_{c},

$$\cos \vartheta_{\text{c}} = \left(\frac{3^3}{2^5}\right)^{1/2}; \qquad \vartheta_{\text{c}} = 23^\circ\, 17'\, 1''.43. \tag{3.16c}$$

After that Z increases with increasing S until for the extreme Kerr ergosphere it approaches a final value

$$Z_f = \tfrac{3}{4}\sqrt{3} m = (\tfrac{3}{2})^{3/2} m_{\text{ir}} \tag{3.17a}$$

at

$$r = \tfrac{3}{2}\sqrt{2} m_{\text{ir}} = \tfrac{2}{3}\sqrt{3} Z_{\text{f}} \tag{3.17b}$$

and an angular separation of 30°, in accordance with the end of § 3.1. The discussion of the case $\vartheta > \frac{1}{2}\pi$ ($Z < 0$) is analogous, and in this case the angular separations ϑ_{c} and 30° are replaced by $180^\circ - \vartheta_{\text{c}}$ and 150°, respectively.

3.3. *The behaviour of R*

Using equations (2.12) and (3.1) we verify that

$$\dot{R} > 0 \tag{3.18}$$

always and so for the maximal possible S, namely for the extreme Kerr ergosphere, R attains its maximal value

$$R_{\text{max}} = 2 m_{\text{ir}}. \tag{3.19}$$

4. The linear dimensions of the ergosphere's boundaries

The results of the previous section permit the study of the reversible changes of the ergosphere's linear dimensions. In the Kerr coordinates we can use the quantities

$$\mathscr{Z} = 2|Z| \qquad \text{and} \qquad \mathscr{R} = 2R \tag{4.1}$$

to describe the linear dimensions of the ergosphere parallel and perpendicular to the symmetry axis, respectively.

From equations (3.11) and (3.12) (bulge on the axis) and from equations (3.16*a*) and (3.17*a*) (bulge off the axis) we find

$$2 \geqslant \mathscr{Z}/2m_{\text{ir}} \geqslant \sqrt{3} \tag{4.2}$$

and

$$3^3/2^5 \leqslant \frac{\mathscr{Z}}{2m_{\text{ir}}} \leqslant (\tfrac{3}{2})^{3/2} \tag{4.3}$$

while equations (3.20) and (4.1) imply

$$\mathscr{R}_{\max}/2m_{\text{ir}} = 2. \tag{4.5}$$

Therefore during the reversible increase of S the maximal (for every $S < 2m_{\text{ir}}^2$) linear dimension parallel to the symmetry plane (always occurring on it) increases continuously and attains its maximal possible value in the case of the extreme Kerr ergosphere. However, the maximal linear dimension parallel to the rotation axis does not change uniformly with increasing S, being maximal for $S = 0$ and minimal for $S = 2m_{\text{ir}}^2$.

It has to be emphasised that in order to reach these conclusions we expressed all the distances in units of $2m_{\text{ir}}$. This unit is always preferable, compared with the total mass energy, in view of its constancy during the reversible transformations of the ergosphere.

In figure 1 we have plotted the cross sections of the Kerr ergosphere's boundaries for $S = mm_{\text{ir}}$, $S = \sqrt{2}m_{\text{ir}}^2$, $S = 2m_{\text{ir}}^2$.

5. Discussion and outlook

The conclusions of this research can be summarised as follows:

(i) The shape of the stationary-limit surface of a Kerr ergosphere depends in a crucial way on the relative value of S/m and m_{ir}.

(ii) A bulge always forms on the ergosphere's outer boundary, either on the rotation axis, if $S/m < m_{\text{ir}}$, or off it, if $S/m > m_{\text{ir}}$.

(iii) During the reversible transformations of the ergosphere its shape changes and the bulge can either approach the symmetry plane, if $0 < S < mm_{\text{ir}}$ and $mm_{\text{ir}} < S < \sqrt{2}m_{\text{ir}}^2$, or recede from it, if $\sqrt{2}m_{\text{ir}}^2 < S < 2m_{\text{ir}}^2$.

(iv) The angular separation of the bulge from the rotation axis $z > 0$ (< 0) always increases (decreases) with reversibly increasing S, approaching its maximal (minimal) value of 30° (150°) for the extreme Kerr ergosphere.

(v) The linear dimension of a given ergosphere parallel to the symmetry plane is maximal on it, and increases uniformly, as S increases reversibly, approaching an upper value of $4m_{\text{ir}}$ for the extreme Kerr ergosphere. The linear dimension parallel

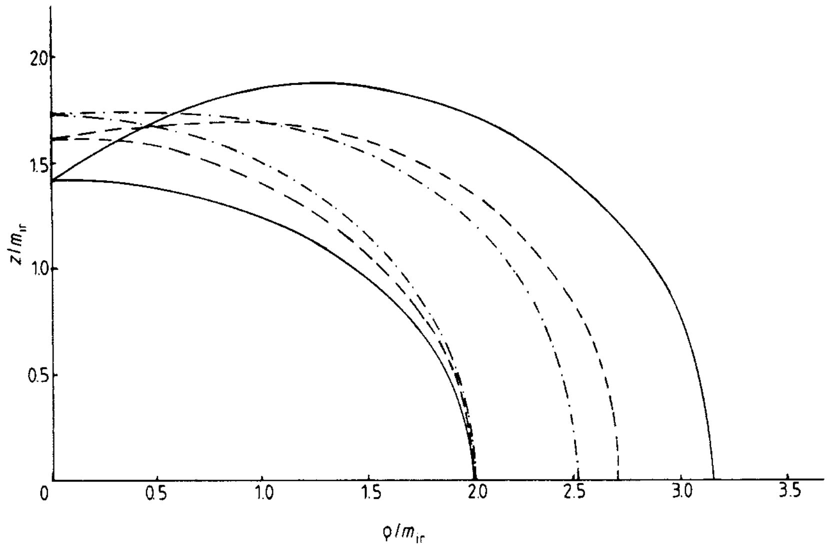

Figure 1. The cross sections of the Kerr ergosphere's boundaries are shown for $S = mm_{\rm ir}$ (*chain curves*), $S = \sqrt{2}m_{\rm ir}^2$ (*broken curve*) and $S = 2m_{\rm ir}^2$ (*full curve*). In each case from the two curves shown the upper curve corresponds to the stationary-limit surface and the lower line to the event horizon. The last case corresponds to the extreme Kerr ergosphere. The change of the ergosphere's shape with reversibly increasing S is obvious.

to the rotation axis does not change uniformly. For $z > 0$ its maximal value is $4m_{\rm ir}$, occurring on the axis for vanishing S, while its minimal value is $\frac{27}{16}m_{\rm ir}$, occurring for $S = \sqrt{2}m_{\rm ir}^2$ (namely, *before* the extreme state of the Kerr ergosphere is reached) off the axis at an angular separation of 23°.28 from it.

To the extent of our knowledge, the content of the present article constitutes the first effort for a complete description of the shape and size of an uncharged ergosphere along with its reversible transformations. The method applied is coordinate dependent, but the conclusions concerning the conditions for the formation of the bulge and its behaviour during the reversible transformations are quite general and coordinate independent.

As far as the relation is concerned of our results and the ergosphere formed external to a slowly rotating material configuration, we recall that, according to Chandrasekhar and Miller (1974) such homogeneous fluid configurations might develop an ergosphere near the equatorial plane. So, in this slow-rotation case the bulge does not form, in the sense that it always remains (rather) close to the rotation axis.

The shape of the Kerr ergosphere's boundaries has been considered by other authors. Thus, Sharp (1981) examining the embeddings of the Kerr geometry proved that for $S < m^2$ the slope $dz/d\rho$ of the stationary-limit surface vanishes either on the rotation axis ($\vartheta = 0, \pi$) or off it, at ϑ defined exactly by equation (2.8) in the text, while for $S = m^2$ the slope vanishes only at ϑ defined by equation (2.8), namely, at an angular separation 30° (or 150°). Obviously this is exactly the behaviour of z' and this simply means that the vanishing of $dz/d\rho$ $(= z'/\rho')$ is a consequence of the condition $z' = 0$. However, Sharp's results (and figures), although generally in accordance with figure 1, have been based on the interpretation of r and ϑ as polar (not quasi-spheroidal) coordinates in flat space–time, as a consequence of which the ergosphere's shape is not exactly the same as in figure 1. Thus, the surface of constant r (e.g. the event horizon) is a sphere of radius r and not an ellipsoid of revolution. Moreover, the slope of the stationary-limit surface is different. Thus, for the extreme Kerr ergosphere this slope on the symmetry axis is $\pm 2^{-1/2}$, the corresponding result of Sharp being ± 1. On the other hand, the apparent discontinuity of the slope on the rotation axis of figure 1 is simply, as Sharp also remarked, a manifestation of the inner stationary-limit surface

$$r = R_{-} \equiv m\left[1 - \left(1 - \frac{S^2}{m^4}\cos^2\vartheta\right)^{1/2}\right]$$

which joins smoothly with the outer one $r = R_{+}$. It has to be emphasised at the outcome that Sharp did not examine the relation between the ergosphere's shape (and dimensions) and its reversible transformations.

The last argument is also true for the results of Hoenselaers (1980) referring to more general stationary, axisymmetric and asymptotically flat, vacuum solutions of the Einstein's equations. It has to be pointed out, moreover, that according to these results, obtained with the aid of the Weyl coordinates, the stationary-limit surface can possibly have no common points with the plane $z = 0$, except at the origin. This, in view of the results of Chandrasekhar and Miller (1974), implies that at least some of the solutions of Hoenselaers (1980) cannot describe ergospheres formed external to (slowly) rotating material configurations.

Moreover Bardeen (1973) derived the *apparent* shape and size of the Kerr ergosphere, based on the distant observations of the photon's trajectories in the hole's vicinity. Contrasted with ours, his results are characterised by the shape's distortion, a decrease of Θ by about 9°, and an increase in $\mathscr{X}$ and $\mathscr{R}_{\max}$ by factors of 8 and $\sqrt{2}$, respectively, due to the bending of light rays along with the frame-dragging effect induced by the hole's rotation.

Finally we should point out that the present article deals with the more astrophysical but less general, uncharged ergosphere. The case of the more general, charged (Kerr–Newmann) ergosphere, as well as the time scale of the ergosphere's surface transformations, depending on the specific way that S increases, along with their probable astrophysical applications will be dealt with in a forthcoming article.

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References

Bardeen J M in *Black Holes* 1973 ed C DeWitt and B S DeWitt (New York: Gordon and Breach) pp 215–39
Boyer R H and Lindquist R W 1967 *J. Math. Phys.* **8** 265
Chandrasekhar S and Miller J C 1974 *Mon. Not. R. Astron. Soc.* **167** 63
Christodoulou D 1970 *Phys. Rev. Lett.* **25** 1596
Hoenselaers C 1980 *J. Math. Phys.* **21** 2241
Sharp N A 1981 *Can. J. Phys.* **59** 688